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A NOTE ON CHROMATICITY OF MULTIBRIDGE GRAPHS

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ABSTRACT

Coloring the vertices of graph such that every two adjacent vertices have different colors is a very common problem in graph theory. This is called a proper coloring of the graph. The possible number of different proper colorings on a graph with a given number of colors can be represented by a function called the chromatic polynomial. Two graphs G and H are said to be chromatically equivalent, if they share same chromatic polynomial. A graph G is chromatically unique, if G isomorphic H for any graph H such that G is chromatically equivalent H. The study of chromatically equivalent and chromatically unique problems called chromaticity. In this paper, we prove the chromatic uniqueness of a new family of multibridge graphs.

KEYWORDS: Graph & Chromatic Polynomial

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Original Article

1. INTRODUCTION

The graphs that we consider here are finite, undirected and simple. Let P (G, λ) denote the chromatic polynomial of a graph G. Two graphs *G* and *H* are said to be chromatically equivalent, and we write *G* ~ H, if P (G, λ) = P (H, λ). A graph *G* is chromatically unique, unique if $G \cong H$ for any graph *H* such that *G* ~ *H*. By subdivision we mean the operation of replacing an edge of a graph by a path. If a graph *H* can be derived from *G* by a sequence of subdivisions, we say *H* is a subdivision of *G*. For each positive integer *f*, the graph *G(f)* obtained from *G* by replacing each edge of *G* with a path of length *f* is called the *f* -uniform subdivision of *G*. The operation replaces a *u-v* chain by an edge *uv* called chain-contraction. By contracting all maximal chains of a graph G, we arrive at multigraph *M (G)*. Two graphs *G* and *H* are homeomorphic if *M (G)* = *M (H)*. If *G* is homeomorphic to *H* we also say *G* is a *H*-homeomorph.

For each integer $k \geq 2$, let θ_k be the multigraph with two vertices and k edges. Any subdivision of θ_k is called multi-bridge graph or k-bridge graph. We denote $\theta\ (a_1, a_2,..., a_k)$ where $a_1, a_2,..., a_k \in N$ to be the graph obtained by replacing the edges of θ_k by paths of length $a_1, a_2,..., a_k$ respectively. Kalaf [6], Khalaf and Peng [7-11] showed the chromatic uniqueness of several kinds of k-bridge graphs.

We will solve a problem stated by Dong, Koh and Teo [4] related to chromaticity of 5-bridge graphs.

2. KNOWN RESULTS

In this section we cite some results previous results related to our objectives. A 2-bridge graph is simply cycle, which is χ – unique [4]. Chao and whithead Jr. [2], showed that every 3-bridge graph θ (1, a2, a3) called a theta graph is χ –unique. Loerinc [15] extended the above result to all 3-bridge graphs (called generalized θ graph). Bao and Chen [1] showed that every 5-bridge graph is χ − unique if its shortest maximal chains of length

greater than three. This result is special case of general result due to Xu, Liu and Peng [17].

Theorem 1 [17] For k ≥ 4, θ (a1, a2,..., ak) is χ – unique if k – 1 ≤ a1 ≤ a2 ≤ · · ·≤ ak.

Li and Wei [14] established the following result:

Theorem 2 [14] For a, b ∈ N, where b ≥ a ≥ 3, the 5-bridge graph θ (2, 2, 2, a, b) is χ – unique if and only if (a,b)=(3,4).

Ye [18] extended the above result to any k-bridge graph θ (2, 2, · · ·, a, b) with b≥ a ≥ 3 and k ≥ 5. Li and Wei [14] stated without proof that θ(a, a, a, b, c), where c ≥ b ≥ a + 1 ≥ 3, is χ – unique if and only if (b, c) = (a + 1, a + 2). Xu, Liu, and Peng [17] showed that any f-uniform subdivision of θk is χ − unique, as in this theorem:

Theorem 3 [17] For k ≥ 2, the graph θk(f) is χ − unique.

The above result was proved independently by Dong [3], Koh and Teo [12] and Xu, Liu and Peng [17]. Peng [16] proved the following result:

Theorem 4 [16] Let h, s ∈ N such that h ≥ s + 1 ≥ 2 or s = h + 1. Then for k ≥ 4, θ(a1, a2,..., ak) is χ − unique if a2-1=a1=h, aj=h+s (j=3,…,k-1), ak≥ h+s and $a_k \notin \{2h, 2h + s, 2h + s - 1\}$. Dong, Teo, Little, Hendy and Koh [5] proved this result:

Theorem 5 [5] If 2 ≤ a1 ≤ a2 ≤ · · ·≤ ak < a1 + a2, where k ≥ 3, then the graph θ(a1, a2,..., ak) is χ − unique.

Let $a_1, a_2, \dots a_k \in \mathbb{N}$, and $G = \theta(a_1, a_2, \dots a_k)$. Then see [4]:

$$P(G,\lambda) = \frac{1}{\lambda^{k-1}(\lambda-1)^{k-1}} \prod_{i=1}^{k}\left((\lambda-1)^{a_i+1} + (-1)^{a_i+1}(\lambda-1)\right) + \frac{1}{\lambda^{k-1}} \prod_{i=1}^{k}((\lambda-1)^{a_i} + (-1)^{a_i}(\lambda-1)).$$

Let $\lambda = 1 - x$, then:

$$P(G, 1-x) = \frac{(-1)^{a_1,a_2,\dots a_k+1}}{(1-x)^{k-1}}\left(x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)\right)$$

$$= \frac{(-1)^{e(G)+1}}{(1-x)^{e(G)-v(G)+1}}\left(x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x)\right),$$

where $e(G) = \sum_{i=1}^{k} a_i$ *and* $v(G) = \sum_{i=1}^{k} a_i - k + 2$.

Also they defined Q (G, x) for any graph G and real number x as:

$Q(G,x) = (-1)^{e(G)+1}(1-x)^{e(G)-v(G)+1}p(G, 1-x)$, and they got the following results:

Theorem 6 [4] for any k,a_1, a_2, a_k ∈ N

$$Q(\theta(a_1, a_2, \dots, a_k, x) = x\prod_{i=1}^{k}(x^{a_i}-1) - \prod_{i=1}^{k}(x^{a_i}-x).$$

In particular, $Q(C_n, x) = x(x-1)(x^{n-1}-1)$.

Theorem 7 [4] for any tow graphs *G* and *H*,

$(i) if\ H \sim G, then\ Q(H,x) = Q(G,x);$

$(ii) if\ Q(H,x) = Q(G,x) and\ v(H) = v(G), then\ H \sim G.$

Lemma 8 [4] suppose that $\theta(a_1, a_2, \dots, a_k) \sim \theta(b_1, b_2, \dots, b_k)$,

Where $k \geq 3, 2 \leq a_1 \leq a_2 \leq \cdots \leq 2 \leq b_1 \leq b_2 \leq \cdots \leq b_k$. then $a_i = b_i$ for all *i=1,2,…,k*.

Dong et al denote $g_e(G_1, G_2, \dots, G_k)$to the collection of all edge-gluings of all $G_1, G_2, \dots, G_k, wherer\ k \geq 2\ and\ e(G_i) \geq 1$ for all i, and then they got the following lemma:

Lemma 9 [4] Let $H \sim \theta(a_1, a_2, \dots, a_k), where\ k \geq 3\ and\ a_i \geq 2$ for all i.

Then one of the following is true:

1. $H \cong \theta(a_1, a_2, \dots, a_k)$;

2. $H \in g_e\left(\theta(b_1, b_2, \dots, b_t), C_{b_{t+1}} + 1, \dots C_{b_{k+1}}\right)$, Where $3 \leq t \leq k-1\ and\ b_i \geq 2$ for all *i=1,2,…,k*.

Theorem 10 [4] Let $k, t, b_1, b_2, \dots, b_k \in \mathrm{N}\ with\ 3 \leq t \leq k-1\ and\ b_i \geq 2$ for all *i=1,2,…,k*.

If $H \in g_e(\left(\theta(b_1, b_2, \dots, b_t), C_{b_{t+1}} + 1, \dots C_{b_{k+1}}\right)$ then

$Q(H, x) = x \prod_{i=1}^{k}(x^{b_i} - 1) - \prod_{i=t+1}^{t}(x^{b_i} - x) \prod_{i=t+1}^{k}(x^{a_i} - 1)$.

Dong, Teo, Little, Hendy and Koh (2004) showed that the graph $\theta(2{,}2{,}2{,}3{,}4)$

Theorem 11 [5] $\theta\ (2, 2, 2, 3, 4) \sim H\ for\ every\ H\ the\ family\ g_e\theta(2{,}2{,}2{,}3), C_4, C_4)$.

3. RESULTS

In this section we solve a problem stated Dong, Koh and Teo [4] and give another result related to 6-bridge graph.

Problem 12 [4] Study the chromaticity of $\theta(a, a_2, \dots, a_5), where\ 2 \leq a_1 \leq a_2 \leq \cdots \leq a_5 \leq 3$.

We completely solve the above problem by following proposition:

Proposition 13 The 5-bridge graph $\theta(a, a_2, \dots, a_5), where\ 2 \leq a_1 \leq a_2 \leq \cdots \leq a_5 \leq 3$ is χ-unique.

Proof: There are six cases to be considered, depending on the length of a_i, $1 \leq i \leq 5$.

Case 1: The graph $\theta(a, a_2, \dots, a_5) = \theta(2{,}2{,}2{,}2{,}2)$, that is the graph $\theta_5(2)$ is x- unique by theorem 3.

Case 2: The graph $\theta(a, a_2, \dots, a_5) = \theta(2{,}2{,}2{,}2{,}3)$ is x- unique because this graph is one of x- unique graphs listed by Li [13].

Case 3: The graph $\theta(a, a_2, \dots, a_5) = \theta(2{,}2{,}2{,}3{,}3)$ is χ- unique because (a, b)=(3,3)≠(3,4) in the theorem 2.

Case 4: The graph $\theta(a, a_2, \dots, a_5) = \theta(2{,}2{,}3{,}3{,}3)$. In this graph $2 \leq a_1 \leq a_2 \leq \cdots a_5 \leq a_1 + a_2 = 4$.

By Theorem 5 is χ- unique.

Case 5: The graph $\theta(a_1, a_2, \dots, a_5) = \theta(2{,}3{,}3{,}3{,}3)$ choose h=2 and s=1, we see that h≥s+1=2 and a_1 =2, a_2 =3, a_3 =3, a_4 =3, a_5 =3∉{2h,2h+s, 2h+s-1}. By Theorem 4, we conclude that$(2{,}3{,}3{,}3{,}3)$ is χ- unique.

Case 6: The graph $\theta(a, a_2, \dots, a_5) = \theta(3{,}3{,}3{,}3{,}3)$ that is graph $\theta_5(3)$ which is - unique By Theorem 3. ∎

Lemma 14 Let $H \sim\ \theta(a_1, a_2, \dots, a_5), if\ a_1 > 2, a_2 \leq a_2 \leq \cdots \leq a_5$. Then one of the following is true:

$(i) H \sim \theta(a_1, a_2, \dots, a_5)$;

$(ii) H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5} + 1\big)$, where $b_i \geq 2$ for all i=1,2,…,5.

Proof: Let $H \sim \theta(a_1, a_2, \dots, a_5)$ then by Lemma 2 of the following is true :

$(i) H \sim \theta(a_1, a_2, \dots, a_5)$;

$(ii) H \in g_e\big(\theta(b_1, b_2, \dots, b_t), C_{b_{t+1}} + 1, \dots, C_{b_{5+1}}\big)$, where $3 \leq t \leq 4$ and $b_i \geq 2$ for all *i=1,2,…,5*

Now,

$$Q(G, x) = x \prod_{i=1}^{k} (x^{a_i} - 1) - \prod_{i=1}^{t} (x^{a_i} - x)$$

$$=-x^{a_1+1+a_2} + x^{1+a_2} - x^{a_3+4} - x^{4+a_4} + x^{3+a_4+a_5} - x^{a_5+1+a_2} - x^{a_3+2+a_1+a_2} - x^{a_3+a_4+a_5+a_1+a_2} - x^{2+a_4+a_1+a_2} + x^{3+a_5+a_1} + x^{a_3+} + x^{a_3+1} + x^{1+a_3+a_4+a_5+a_1+a_2} + x^{a_3+a_4+1+a_1} + x^{1+a_3+a_4+a_2} - x^{a_5+2+a_1+a_2} - x^{2+a_4+a_5+a_2} + x^{a_3+a_4+1} + x^{a_3+1+a_5+a_1} + x^5 + x^{3+a_4+a_1} - x^{2+a_4+a_5+1} + x^{a_1+1} - x^{4+a_5} + x^{a_3+a_4+a_5+1} + x^{a_3+a_4+3} - x^{a_3+a_4+a_5+2} + x^{1+a_3+a_5+a_2} - x^{1+a_4+a_5} + x^{a_3+3+a_2} - x^{a_3+a_4+2+a_2} - x^{a_3+2+a_5+a_2} - x^{a_3+a_4+2} + x^{1+a_4+a_5+a_2} - x^{1+a_3+a_1} - x^{4+a_1} + x^{a_3+3+a_5} + x^{3+a_1+a_2} - x^{1+a_4+a_1} + x^{a_5+a_1+a_2} - x^{1+a_3+a_2} + x^{3+a_5+a_2} +^{x^{a_3+3+a_1} + x^{1+a_3+a_1+a_2} + x^{3+a_4+a_2} - x^{a_3+2+a_5+a_1} - x^{4+a_2} + x^{1+a_4} +} x^{1+a_4+a_1+a_2} - x^{a_5+a_1+1} + x^{1+a_4+a_5+a_1} + x^{a_5+1} - x^{1+a_4+a_2} - x^{a_3+1+a_5} - x$$

and *Q (H, x)* depend on value of t. If *t=3*, then

$$Q(H, x) = x \prod_{i=1}^{5} (x^{b_i} - 1) - \prod_{i=1}^{3} (b_i - x) \prod_{i=t+1}^{k} (x^{a_i} - 1)$$

$$=x^{2+b_3+b_5} + x^{2+b_1+b_5} - x^{1+b_4+b_2} + x^{b_2+2+b_4} + x^{b_3+b_5+b_1+b_2} - x - x^{b_3+b_1+b_2} - x^{b_5+1+b_1} - x^{2+b_1} + x^{2+b_3+b_4} + x^{b_4+1} + x^{1+b_4+b_5+b_1} - x^{2+b_3+b_4+b_5} - x^{2+b_4+b_5+b_1} - x^{3+b_5} + x^{1+b_3+b_4+b_5+b_1+b_2} + x^{2+b_4+b_1} + x^{1+b_1} - x^{2+b_3} + x^{b_3+b_4+b_1+b_2} + x^{b_2+1} + x^{1+b_3+b_4+b_5} + x^{b_3+b_2+1+b_1} + x^{b_2+2+b_5} - x^{1+b_3+b_4+b_1+b_2} - x^{1+b_3+b_5} - x^{1+b_3+b_4} + x^{b_5+1} - x^{3+b_4} + x^3 + x^{3+b_4+b_5} - x^{b_3+b_2+1+b_1} - x^{b_2+2} + x^{1+b_3} + x^{b_1+1+b_4+b_5} - x^{b_4+b_5+1} - x^{b_2+2+b_4+b_5} - x^{b_3+b_4+b_5+b_1+b_2} - x^{1+b_4+b_1} - x^{1+b_5+b_2}$$

If t=4, then

$$Q(H, x) = x \prod_{i=1}^{5} (x^{b_i} - 1) - \prod_{i=t+1}^{3} (x^{b_5} - 1) \prod_{i=1}^{4} (x^{b_i} - x)$$

$$= -x^{1+b_4+b_2} + x^{b_2+2+b_4} - x^{3+b_3} - x - x^{b_5+1+b_1} + x^{2+b_3+b_4} + x^{b_4+1} + x^{1+b_4+b_5+b_1} - x^{2+b_3+b_4+b_5} - x^{2+b_4+b_5+b_1} + x^{1+b_3+b_4+b_5+b_1+b_2} + x^{2+b_4+b_1} + x^{1+b_1} + x^{b_3+b_4+b_1+b_2} + x^{b_1+1} + x^{1+b_3+b_4+b_5} - x^{1+b_3+b_4+b_1+b_2} - x^{1+b_3+b_5} - x^{1+b_3+b_4} + x^{b_5+1} - x^{3+b_4} + x^{3+b_4+b_5} + x^{1+b_3} + x^{b_2+1+b_4+b_5} - x^{b_4+b_5+1} - x^{b_2+2+b_4+b_5} - x^{b_3+b_4+b_5+b_1+b_2} - x^{1+b_4+b_1} - x^{1+b_5+b_2} - x^{b_3+b_5+b_2+2} + x^4 + x^{1+b_5+b_1+b_2} - x^{4+b_5} - x^{b_2+2+b_5+b_1} + x^{3+b_5+b_1} + x^{2+b_3+b_1} - x^{3+b_3+b_5} - x^{3+b_1} + x^{b_2+2+b_1} - x^{1+b_3+b_1} + x^{b_2+3+b_5} - x^{b_3+b_2+1} - x^{b_2+3} + x^{b_3+b_2+2} - x^{1+b_1+b_2} + x^{1+b_3+b_5+b_1} + x^{b_3+b_2+1+b_5}$$

It is easy to see that if *a>2* and *t=3*, then the term x^3 cannot be cancel with any other terms in *q(H,x)* and *q(G,x)*. Therefore either $H \cong \theta(a_1, a_2, \dots a_5)$ $or H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5} + 1\big)$.

Lemma 15 $if\ a_1 > 3, then\ \theta(a_1, a_2, \dots a_5)$ is χ-unique.

Proof Let $H \sim \theta(a_1, a_2, \dots, a_5)$. By lemma 3, either $H \cong \theta(a_1, a_2, \dots a_5)$ $or H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5} + 1\big)$.

If $H \in g_e\big(\theta(b_1, b_2, b_3, b_4), C_{b_5} + 1\big)$, then

$$Q(H,x) = -x^{1+b_4+b_2} + x^{b_2+2+b_4} - x^{3+b_3} - x - x^{b_5+1+b_1} + x^{2+b_3+b_4} + x^{b_4+1} + x^{1+b_4+b_5+b_1} - x^{2+b_3+b_4+b_5} - x^{2+b_4+b_5+b_1} + x^{1+b_3+b_4+b_5+b_1+b_2} + x^{2+b_4+b_1} + x^{1+b_1} + x^{b_3+b_4+b_1+b_2} + x^{b_1+1} + x^{1+b_3+b_4+b_5} - x^{1+b_3+b_4+b_1+b_2} - x^{1+b_3+b_5} - x^{1+b_3+b_4} + x^{b_5+1} - x^{3+b_4} + x^{3+b_4+b_5} + x^{1+b_3} + x^{b_2+1+b_4+b_5} - x^{b_4+b_5+1} - x^{b_2+2+b_4+b_5} - x^{b_3+b_4+b_5+b_1+b_2} - x^{1+b_4+b_1} - x^{1+b_5+b_2} - x^{b_3+b_5+b_2+2} + x^4 + x^{1+b_5+b_1+b_2} - x^{4+b_5} - x^{b_2+2+b_5+b_1} + x^{3+b_5+b_1} + x^{2+b_3+b_1} - x^{3+b_3+b_5} - x^{3+b_1} + x^{b_2+2+b_1} - x^{1+b_3+b_1} + x^{b_2+3+b_5} - x^{b_3+b_2+1} - x^{b_2+3} + x^{b_3+b_2+2} - x^{1+b_1+b_2} + x^{1+b_3+b_5+b_1} + x^{b_3+b_2+1+b_5}$$

And *Q (G, x)* as in Lemma 3.

The term x_4 in *Q (H, x)* cannot be cancel with any other terms in *Q (H, x)* or in *Q (G, x)*. Contradiction with Theorem 7.

In the same technique using in the above lemma, we prove the following result:

Theorem 16: The 6-bridge graph θ *(a, a, a, a, b, c)*, a ≤ b ≤ c is chromatically unique.

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